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Minimization of Vibration in Elastic Beams with Time-Variant Boundary Conditions

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This paper presents an innovative method for minimizing the vibration of structures with time-variant boundary conditions (supports). The elastic body is modeled in two ways: 1) the first model is a letter seven type beam with a movable mass not to exceed the lower tip; 2) the second model has an arm that is a hollow beam with an inside mass with adjustable position. The complete solutions to both problems are carried out where the body is undergoing large rotation. The quasistatic procedure is used for the time-variant boundary conditions. The method developed employs partial differential equations governing the motion of the beam, including the effects of rigid-body motion, time-variant boundary conditions, and calculus of variations. The analytical solution is developed using Laplace and Fourier transforms. Examples of elastic robotic arms are given to illustrate the effectiveness of the methods developed.

Introduction

THE study of complex interconnected mechanical systems with rigid and flexible articulated components has been a growing interest to scientists and engineers in the past decade. The rich history of the subject derives primarily from the work of mechanism designers and the work of aerospace engineers interested in the modeling and control of complex multibody robotic or spacecraft systems.

The exploitation of the special structures of mechanical system is important in engineering applications. The dynamics and control of interconnected rigid/flexible bodies such as robotic, aeronautic, and space structures is based on mathematical modeling and analysis and their concurrent numerical solutions.[1-8] In modeling rotating systems with continuum-mechanical components, nonlinear models display a behavior that is in certain cases qualitatively quite different from that observed in linear and semilinearized models.[9-13] For instance, if we view a beam equation of, say, an Euler-Bernoulli type as an approximation of a geometrically exact model, then the processes of attachment to a rapidly rotating rigid body and approximation do not commute. Therefore, proper attention to dynamic modeling is a crucial step.

The extension of interconnected flexible multibody to include the effects of flexible tracks for the case of interbody translation is treated in Refs. 14–18. While the later provided an insight into the contribution of the flexible tracks to the dynamic equations, the flexibility effects of the translating body remains one where the modal vectors were treated as invariant. Reference 17 presents a finite element procedure with time-variant mode shapes, and Ref. 18 develops a comprehensive algorithmic procedure for handling constraints resulting from the flexible transmissions/gears in multibody systems. In our paper the concept of the support is modeled through boundary conditions.

The minimization of deformation of the structures in mechanical systems is a major concern in dynamics and control. In robotics, aeronautic, and space structures, lighter weight and higher strength materials with sufficient stiffness are needed. In fact, this new composition of materials requires that the dynamics by coupled with design of optimization techniques. Early studies by Hoppmann,[19-21] Mindlin and Goodman,[22] Young,[23] and Nothmann,[24] suggest that time-variant boundary conditions could be useful in controlling vibration of the beam. References 25–28 presented vibration control approaches for beams with actuators/sensors.

In this paper a movable support (or mass block) is used to model a flexible-beam model of a robotic manipulator arm to minimize the lateral deformation (vibration) of the beam. The rotating inertia force is considered as an external applied force. Laplace and Fourier transforms are used to obtain an analytic solution governing the deformation for the problem. The function of motion of the movable support (or mass block) is obtained through a step-by-step approach using calculus of variations.

The paper is divided into four sections. The first section presents the equivalent model of a system with a terminal flexible link. The equations of motion form the second section. The third section follows with the equations of motion for the second model. Illustrative examples for the second model are presented in the fourth section.

Equivalent System

The system shown in Fig. 1 is a model of a robotic manipulator used to carry a load for a specific high-speed operation. The dashed structure and the controllable ball (moving support) are added to minimize the vibration of beam l, which is assumed to be slender and flexible. For convenience we assume that the links (bodies) l_1, l_2, l_3, and l_4 have larger stiffness and can be considered as rigid bodies. Then we obtain an equiva-

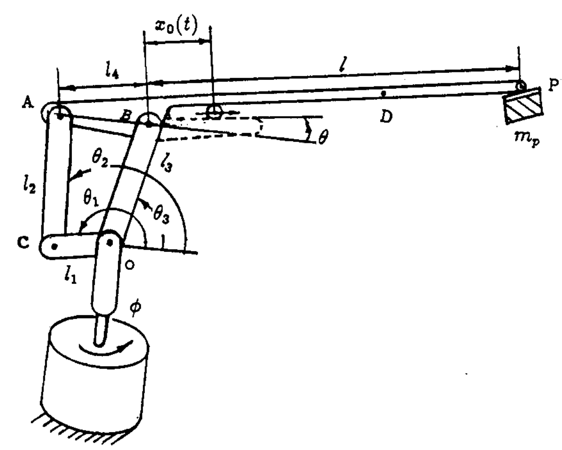

Fig. 1 Robotic manipulator with a flexible terminal link.

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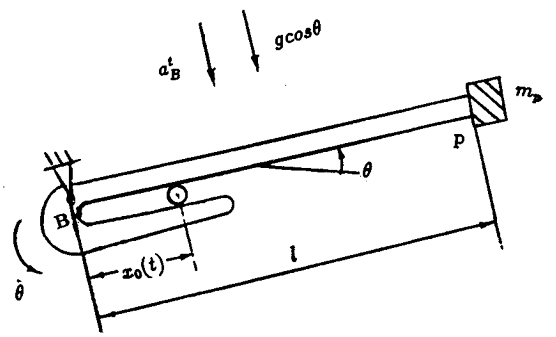

Fig. 2 Equivalent system model to Fig. 1.

lent system depicted in Fig. 2, which takes into consideration the relative rigid-body motion. It should be assumed further that in the developed model the bending in the vertical plane is independent of the rotation ϕ. The acceleration relationship between the arbitrary point D and the joint point B is

$$\bar{a}_D = \bar{a}_B + \bar{a}_{DB} = \bar{a}_B^n + \bar{a}_B^t + \bar{a}_{DB}^n + \bar{a}_{DB}^t \tag{1}$$

where $\bar{a}_{DB}$ is the relative acceleration between points B and D, n indicates normal (in the x direction), and t indicates tangential (in the y direction). Since the normal acceleration does not affect the transverse vibration in this case, and

$$a_{DB}^t = l_{DB}\ddot{\theta}$$

where l_{DB} is the length between points B and D, we can get a relative motion system that has a translation acceleration a_B^t, as shown in Fig. 2.

Let $\bar{l}_1 + \bar{l}_2 + \bar{l}_4 = \bar{l}_3 = \bar{p}_B$, where $\bar{l}_i\ (i = 1, \ldots, 4)$ is the vector of the body i and $\bar{p}_B$ is the vector of point B. Then we obtain

$$l_1\cos\theta_1 + l_2\cos\theta_2 + l_4\cos\theta = l_3\cos\theta_3 \tag{2}$$

$$l_1\sin\theta_1 + l_2\sin\theta_2 + l_4\sin\theta = l_3\sin\theta_3 \tag{3}$$

or

$$\cos\theta(l_2\cos\theta_2 + l_1\cos\theta_1) + \sin\theta(l_2\sin\theta_2 + l_1\sin\theta_1) = C \tag{4}$$

where

$$C = -\frac{1}{2l_4}\left[l_3^2 - l_4^2 - l_2^2 - l_1^2 - 2l_1l_2\cos(\theta_2 - \theta_1)\right] \tag{5}$$

and $\theta_i\ (i = 1, 2, 3)$ and θ are shown in Fig. 1. From this equation we derive

$$\theta = \sin^{-1}\frac{C}{\sqrt{A_2 + B^2}} - \tan^{-1}\frac{A}{B} \tag{6}$$

where

$$A = l_2\cos\theta_2 + l_1\cos\theta_1 \tag{7}$$

$$B = l_2\sin\theta_2 + l_1\sin\theta_1 \tag{8}$$

The second differentiation of Eq. (6) yields

$$\ddot{\theta} = \frac{\ddot{E}}{\sqrt{1 - E^2}} + \frac{E\dot{E}^2}{(1 - E^2)^{3/2}} - \frac{\ddot{D}}{1 + D^2} - \frac{2D\dot{D}}{(1 + D^2)^2} \tag{9}$$

where $E = C/\sqrt{A^2 + B^2}$ and $D = A/B$.

The position of point B is given by the components of vector $\bar{p}_B$:

$$\{p_B\} = \begin{bmatrix} l_3\cos\theta_3 \\ l_3\sin\theta_3 \end{bmatrix} \tag{10}$$

Therefore, the acceleration of point B in the fixed ξ-η coordinate system shown in Fig. 1 is

$$\{a_B\} = \begin{bmatrix} a_B^\xi \\ a_B^\eta \end{bmatrix} = \begin{bmatrix} -l_3(\cos\theta_3\dot{\theta}_3^2 + \sin\theta_3\ddot{\theta}_3) \\ -l_3(\sin\theta_3\dot{\theta}_3^2 - \cos\theta_3\ddot{\theta}_3) \end{bmatrix} \tag{11}$$

The normal and tangential components are obtained as follows:

$$\begin{bmatrix} a_B^n \\ a_B^t \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} a_B^\xi \\ a_B^\eta \end{bmatrix} \tag{12}$$

where we get

$$a_B^t = l_3\sin\theta(\cos\theta_3\dot{\theta}_3^2 + \sin\theta_3\ddot{\theta}_3) - l_3\cos\theta(\sin\theta_3\dot{\theta}_3^2 - \cos\theta_3\ddot{\theta}_3) \tag{13}$$

Obviously, if θ_3 is fixed or equal to constant, then $a_B^t = 0$.

From Eqs. (6), (9), and (13), we obtain the equivalent system shown in Fig. 2.

Equations of Motion

In Fig. 2, $\theta = \theta(\theta_1, \theta_2)$, $0 < x_0(t) < l_0$, and l_0 is a given constant. The objective is to find $x_0(t)$ that minimizes y_p, which is the deformation of point p, thus reducing the structure size and weight, since the use of smaller size beams will undoubtedly not yield larger deformation. For the case $l_0 = l$ (i.e., the ball can reach point p), the objective is, of course, to minimize the vibration of the whole beam.

By using d'Alembert's principle, the governing equation of the beam is given by

$$EI\frac{\partial^4 y}{\partial x^4} = f^*(x, y, t) \tag{14}$$

where $y = y(x, x_0, t)$, E is Young's modulus, I is the moment of inertia of area for bending, and $f^*(x, y, t)$ denotes the generalized force given by

$$f^*(x, y, t) = m\left(y\dot{\theta}^2 - x\ddot{\theta} - \frac{\partial^2 y}{\partial t^2} - g\cos\theta - a_B^t\right) \tag{15}$$

Hence, we obtain

$$EI\frac{\partial^4 y}{\partial x^4} + m\left(\frac{\partial^2 y}{\partial t^2} - y\dot{\theta}^2 + g\cos\theta + a_B^t + x\ddot{\theta}\right) = 0 \tag{16}$$

where $a_B^t = a_B^t(\theta_1, \theta_2)$ is the contribution of relative translation inertia force, $g\cos\theta$ is the contribution of gravity, m is the mass per unit length of the beam, $\ddot{\theta} = \ddot{\theta}(t)$ is the angular acceleration, and $y\dot{\theta}^2$ is the contribution of the centrifugal force due to the deformation in the y direction. If the deformation is small or angular velocity is small, this term is negligible.

The boundary conditions consist of the following.

Essential conditions:

$$y(0, x_0, t) = 0 \tag{17}$$

$$\frac{\partial}{\partial x}y(0, x_0, t) = 0 \tag{18}$$

$$y(x_0, x_0, t) = 0 \tag{19}$$

Natural conditions:

$$\frac{\partial^2}{\partial x^2}y(l, x_0, t) = \frac{M_e}{EI} = 0 \tag{20}$$

$$\frac{\partial^3}{\partial x^3}y(l, x_0, t) = -\frac{Q_e}{EI} = -\frac{m_p}{EI}\left[g\cos\theta + l\ddot{\theta} + a_B^t + \frac{\partial^2}{\partial t^2}y(l, x_0, t)\right] \tag{21}$$

where M_e is an external moment at point p; Q_e is the external shear force caused by the load at point p, in this case; and m_p is the mass of the load, which is assumed to be carried by the robot hand at the end tip of the beam.

The initial conditions consist of the following:

$$y(x,x_0,0)=f_1(x) \tag{22}$$

$$\frac{\partial}{\partial t}y(x,x_0,0)=f_2(x) \tag{23}$$

Solution for the Position of the Mass x_0

Now let us consider x_0 to have a certain value, making use of Laplace transforms, and neglecting the contribution of the centrifugal force. The equation of motion becomes

$$EIY^{(4)}+ms^2Y=L(x,s)+m\left[sf_1(x)+f_2(x)\right] \tag{24}$$

where

$$Y=Y(x,x_0,s)$$

$$L(x,s)=-\mathcal{L}\left\{mg\cos\theta+ma_b^t+mx\ddot{\theta}\right\} \tag{25}$$

where $\mathcal{L}$ denotes the Laplace transform operator.

Equation (24) is a fourth-order ordinary differential equation. Assuming the fourth-order derivative of the initial conditions $f_1^{(4)}=f_2^{(4)}=0$ for simplicity, we find the general solution as follows:

$$Y=\sum_{i=1}^{4}c_ie^{r_ix}+\frac{1}{ms^2}\left\{L(x,s)+m\left[sf_1(x)+f_2(x)\right]\right\} \tag{26}$$

where the complex roots are given by

$$r_1=\beta(1+j)/\sqrt{2} \tag{27}$$

$$r_2=\beta(1-j)/\sqrt{2} \tag{28}$$

$$r_3=\beta(-1-j)/\sqrt{2} \tag{29}$$

$$r_4=\beta(-1+j)/\sqrt{2} \tag{30}$$

$$\beta=(ms^2/EI)^{1/4} \tag{31}$$

and

$$j=\sqrt{-1}$$

In Eq. (26) the first part is the general solution for the homogeneous equation of Eq. (24) and the second part is the particular solution corresponding to the nonhomogeneous part. Let

$$Y=\begin{cases}Y_1 & \text{for} \quad 0<x<x_0\\ Y_2 & \text{for} \quad x_0\le x<l\end{cases} \tag{32}$$

The boundary conditions for Y_1.

$$Y_1|_{x=0}=\frac{\partial}{\partial x}Y_1|_{x=0}=Y_1|_{x=x_0}=0 \tag{33}$$

may be used in Eq. (26) to obtain

$$0=\sum_{i=1}^{4}c_{1i}+\frac{1}{ms^2}\left\{L(0,s)+m\left[sf_1(0)+f_2(0)\right]\right\} \tag{34}$$

$$0=\sum_{i=1}^{4}c_{1i}r_i+\frac{1}{s^2}\left[sf_1'(0)+f_2'(0)-\Theta(s)\right] \tag{35}$$

$$0=\sum_{i=1}^{4}c_{1i}e^{r_ix_0}+\frac{1}{ms^2}\left\{L(x_0,s)+m\left[sf_1(x_0)+f_2(x_0)\right]\right\} \tag{36}$$

where

$$\Theta(s)=\mathcal{L}\{\ddot{\theta}\} \tag{37}$$

Noting that $f_1(0), f_2(0), f_1'(0), f_2'(0), f_1''(l)$, and $f_2''(l)$ are equal to zero from the essential conditions of the beam, the preceding equations reduce to

$$\sum_{i=1}^{4}c_{1i}+\frac{L(0,s)}{ms^2}=0 \tag{38}$$

$$\sum_{i=1}^{4}c_{1i}r_i-\frac{\Theta(s)}{s^2}=0 \tag{39}$$

$$\sum_{i=1}^{4}c_{1i}e^{r_ix_0}+\frac{L(x_0,s)}{ms^2}=0 \tag{40}$$

From these equations we can solve for c_{12}, c_{13}, and c_{14} (the solutions are too tedious to list here) in terms of c_{11}.

The boundary conditions for Y_2 are

$$Y_2|_{x=x_0}=\frac{\partial^2}{\partial x^2}Y_2|_{x=l}=0 \tag{41}$$

$$\frac{\partial^3}{\partial x^3}Y_2|_{x=l}=-\frac{m_p}{EI}\left[G(s)+s^2Y_2|_{x=l}-sf_1(l)-f_2(l)\right] \tag{42}$$

where

$$G(s)=\mathcal{L}\left\{g\cos\theta+l\ddot{\theta}+a_B^t\right\} \tag{43}$$

The preceding equations can be written more explicitly as

$$\sum_{i=1}^{4}c_{2i}e^{r_ix_0}+\frac{L(x_0,s)}{ms^2}=0 \tag{44}$$

$$\sum_{i=1}^{4}c_{2i}r_i^2e^{r_il}=0 \tag{45}$$

$$-\frac{m_p}{EI}\left[G(s)+s^2Y_2|_{x=l}-sf_1(l)-f_2(l)\right]$$

$$=\sum_{i=1}^{4}c_{2i}r_i^3e^{r_il}+\frac{1}{s^2}\left[sf_1'''(l)+f_2'''(l)\right] \tag{46}$$

From Eqs. (44–46) we can solve for c_{22}, c_{23}, and c_{24} in terms of c_{21}, and from equations

$$\frac{\partial}{\partial x}Y_1|_{x=x_0}=\frac{\partial}{\partial x}Y_2|_{x=x_0} \tag{47}$$

$$\frac{\partial}{\partial x^2}Y_1|_{x=x_0}=\frac{\partial^2}{\partial x^2}=Y_2|_{x=x_0} \tag{48}$$

we can get c_{11} and c_{21} uniquely. Now let us use the Fourier transforms to solve the problem. Assume that the initial conditions are zero. If they are not zero, when as $t\to\infty$, their contributions to vibration will vanish because of the internal damping property of the material of the beam.

Applying the Fourier transform to Eq. (16) and letting $f_1(x)=f_2(x)=0$ and $s=j\omega(j=\sqrt{-1})$, we get the general solution to Eq. (16) as

$$Y=c_1\cosh\beta x+c_2\sinh\beta x+c_3\cos\beta x$$

$$+c_4\sin\beta x-\frac{F(x,\omega)}{m\omega^2} \tag{49}$$

where

$$c_i=c_i(x_0,\omega), \qquad i=1,\ldots,4 \tag{50}$$

$$F(x,\omega)=-\mathfrak{F}\left\{mg\cos\theta(t)+ma_B^t(t)+mx\ddot{\theta}(t)\right\} \tag{51}$$

where $\mathfrak{F}$ denotes the Fourier transform operator, and

$$\beta = (m\omega^2/EI)^{1/4} \tag{52}$$

For $0<x<x_0$, let

$$Y_1 = c_{11}\cosh\beta x + c_{12}\sinh\beta x + c_{13}\cos\beta x + c_{14}\sin\beta x - \frac{F(x,\omega)}{m\omega^2} \tag{53}$$

and for $x_0 \le x \le l$,

$$Y_2 = c_{21}\cosh\beta x + c_{22}\sinh\beta x + c_{23}\cos\beta x + c_{24}\sin\beta x - \frac{F(x,\omega)}{m\omega^2} \tag{54}$$

The boundary conditions for Y_1 may then be stated as follows:

$$Y_1|_{x=0} = \frac{\partial}{\partial x} Y_1|_{x=0} = Y_1|_{x=x_0} = 0 \tag{55}$$

i.e.,

$$c_{11} + c_{13} - \left[F(0,\omega)/m\omega^2\right] = 0 \tag{56}$$

$$c_{12} + c_{14} + \left[\Theta(\omega)/\omega^2\right] = 0 \tag{57}$$

$$c_{11}\cosh\beta x_0 + c_{12}\sinh\beta x_0 + c_{13}\cos\beta x_0 + c_{14}\sin\beta x_0 - \frac{F(x_0,\omega)}{m\omega^2} = 0 \tag{58}$$

where $\Theta(\omega)=\mathfrak{F}\{\ddot{\theta}\}$. After some laboratory work we can solve the linear algebraic equation set for c_{12}, c_{13}, and c_{14} in terms of c_{11}, and Y_1, which includes the coefficient c_{11}. For Y_2 we have the following boundary conditions:

$$Y_2|_{x=x_0} = \frac{\partial^2}{\partial x^2} Y_2|_{x=l} = 0 \tag{59}$$

$$\frac{\partial^3}{\partial x^3} Y_2|_{x=l} = \frac{m_p}{EI}\left[G(\omega) - \omega^2 Y_2|_{x=l}\right] \tag{60}$$

where $G(\omega)=\mathfrak{F}\{g\cos\theta + l\ddot{\theta} + a_B^t\}$, or

$$c_{21}\cosh\beta x_0 + c_{22}\sinh\beta x_0 + c_{23}\cos\beta x_0 + c_{24}\sin\beta x_0 - \frac{F(x,\omega)}{m\omega^2} = 0 \tag{61}$$

$$c_{21}\cosh\beta l + c_{22}\sinh\beta l + c_{23}\cos\beta l + c_{24}\sin\beta l = 0 \tag{62}$$

$$-\frac{m_p}{EI}\left[G(\omega) - \omega^2\left(c_{21}\cosh\beta l + c_{22}\sinh\beta l + c_{23}\cos\beta l + c_{24}\sin\beta l - \frac{F(x,\omega)}{m\omega^2}\right)\right] = \beta^3(c_{21}\sinh\beta l + c_{22}\cosh\beta l + c_{23}\sin\beta l + c_{24}\cos\beta l) \tag{63}$$

From Eqs. (61-63) we can get c_{22}, c_{23}, and c_{24} in terms of c_{21}. Then from equations

$$\frac{\partial}{\partial x} Y_1|_{x=x_0} = \frac{\partial}{\partial x} Y_2|_{x=x_0} \tag{64}$$

$$\frac{\partial^2}{\partial x^2} Y_1|_{x=x_0} = \frac{\partial^2}{\partial x^2} Y_2|_{x=x_0} \tag{65}$$

we can find c_{11} and c_{21} uniquely as analytical functions of ω, $\beta(\omega)$, $G(\omega)$, and $F(x,\omega)$. The solution is then found to be

$$Y(x,x_0,\omega) = \begin{cases} Y_1(x,x_0,\omega) & \text{for} \quad 0 \le x < x_0 \\ Y_2(x,x_0,\omega) & \text{for} \quad x_0 \le x \le l \end{cases} \tag{66}$$

The time history of $y(x,x_0,t)$ can be obtained using the inverse Fourier transform for each given x and x_0.

To minimize the deformation of the beam, we have two objectives. The first objective is to minimize the deformation of the beam at its tip. To accomplish this, the following functional is defined:

$$J(x_0) = \int_0^t F(x,x_0,t)\,\mathrm{d}t \tag{67}$$

where

$$F = y_2^2(l,x_0,t) \tag{68}$$

and where $y_2(l,x_0,t)$ is the time history of $Y_2(l,x_0,\omega)$ given by Eq. (66).

The second objective is to minimize the deformation of the beam as a whole. To do this, we seek a value of x_0 that minimizes the functional

$$F = \int_0^l y^2(x,x_0,t)\,\mathrm{d}x \tag{69}$$

The Euler-Lagrange equation from the theory of calculus of variations is

$$\frac{\partial F}{\partial x_0} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F}{\partial \dot{x}_0}\right) = 0 \tag{70}$$

where $\dot{x}_0 = \mathrm{d}x_0/\mathrm{d}t$. Since the function F does not contain $\dot{x}_0$ explicitly, then $\partial F/\partial\dot{x}_0 = 0$. The differential equation becomes an algebraic equation of the following form:

$$\frac{\partial F}{\partial x_0} = 0$$

This equation can be solved numerically using a finite difference gradient or other methods of optimization, step by step, where x_0 is optimized to yield minimum deformation.

An Alternate Method

An alternate method for minimizing the vibration of the beam shown in Fig. 1 consists of the use of a hollow beam with a movable mass block whose mass density per unit length is m_b, as shown in Fig. 3. Again, the objective is to find the displacement $x_0 = x_0(t)$ to minimize the vibration of the beam. The major advantage of this approach is that there is no additional rigid beam support, which has the tendency to increase the weight of the structure.

The governing equation of the system for this particular case can be expressed as

$$EI\frac{\partial^4 y}{\partial x^4} + m\left(\frac{\partial^2 y}{\partial t^2} + g\cos\theta + a_B^t + x\ddot{\theta}\right) = h(x,t) \tag{71}$$

where the generalized external force $h(x,t)$, which is the contribution of the mass block, is

$$h(x,t) = \begin{cases} -m_b\left(\dfrac{\partial^2 y}{\partial t^2} + g\cos\theta + x\ddot{\theta} + a_B^t\right) & \text{for} \quad x_0 - \dfrac{\delta}{2} < x < x_0 + \dfrac{\delta}{2} \\ 0 & \text{otherwise} \end{cases} \tag{72}$$

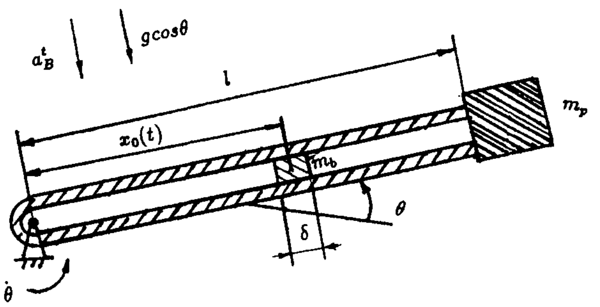

Fig. 3 Alternative method to minimize vibration.

where δ is the length of the mass block, and m_b is the mass per unit length of the mass block. The boundary conditions are as follows.

Essential conditions:

$$y(0,x_0,t)=0 \tag{73}$$

$$\frac{\partial}{\partial x}y(0,x_0,t)=0 \tag{74}$$

Natural conditions:

$$\frac{\partial^2}{\partial x^2}y(l,x_0,t)=0 \tag{75}$$

$$\frac{\partial^3}{\partial x^3}y(l,x_0,t)=-\frac{m_b}{EI}\left[g\cos\theta+l\ddot\theta+a_B^t+\frac{\partial^2}{\partial t^2}y(l,x_0,t)\right] \tag{76}$$

The initial conditions are as follows:

$$y(x,x_0,0)=f_1(x) \tag{77}$$

$$\frac{\partial}{\partial t}y(x,x_0,0)=f_2(x) \tag{78}$$

Following the same procedure used to find the first solution, and using the Laplace transform for Eq. (71), we obtain the following equation:

$$EIY^{(4)}+ms^2Y=H(x,s)+L(x,s)+m\left[sf_1(x)+f_2(x)\right] \tag{79}$$

where

$$H(x,s)=\begin{cases}-m_b\left(s^2Y-sf_1(x)-f_2(x)+\mathcal{L}\{g\cos\theta+x\ddot\theta+a_B^t\}\right) & \text{for}\quad x_0-\dfrac{\delta}{2}<x<x_0+\dfrac{\delta}{2}\\ 0 & \text{otherwise}\end{cases} \tag{80}$$

and $L(x,s)$ is given by Eq. (25). Then by assuming that $f_1^{(4)}(x)=f_2^{(4)}(x)=0$, we get

$$Y=\sum_{i=1}^{4}c_ie^{r_ix}+\frac{1}{ms^2}\left\{H(x,s)+L(x,s)+m\left[sf_1(x)+f_2(x)\right]\right\} \tag{81}$$

where r_i $(i=1,\dots,4)$ are the same as those given by Eq. (29). From the boundary conditions and the conditions that $f_1(0)=f_2(0)=f_1'(0)=f_2'(0)=f_1''(l)=f_2''(l)=0$, we obtain

$$c_1+c_2+c_3+c_4=-\frac{L(0,s)}{ms^2} \tag{82}$$

$$c_1r_1+c_2r_2+c_3r_3+c_4r_4+\frac{\Theta(s)}{s^2}=0 \tag{83}$$

$$c_1r_1^2e^{r_1l}+c_2r_2^2e^{r_2l}+c_3r_3^2e^{r_3l}+c_4r_4^2e^{r_4l}=0 \tag{84}$$

and

$$c_1r_1^3e^{r_1l}+c_2r_2^3e^{r_2l}+c_3r_3^3e^{r_3l}+c_4r_4^3e^{r_4l}=-\frac{m_p}{EI}\left[G(s)+s^2(c_1e^{r_1l}+c_2e^{r_2l}+c_3e^{r_3l}+c_4e^{r_4l})-sf_1(l)-f_2(l)\right]-\frac{1}{s^2}\left[sf_1'''(l)+f_2'''(l)\right] \tag{85}$$

where $\Theta(s)=\mathcal{L}\{\ddot\theta\}$ and $G(s)=\mathcal{L}\{g\cos\theta+l\ddot\theta+a_B^t\}$. Factoring the preceding equations gives

$$\sum_{i=1}^{4}c_i=-\frac{L(0,s)}{ms^2} \tag{86}$$

$$\sum_{i=1}^{4}c_ir_i=\frac{\Theta(s)}{s^2} \tag{87}$$

$$\sum_{i=1}^{4}c_ir_i^2e^{r_il}=0 \tag{88}$$

$$\sum_{i=1}^{4}c_i\left(r_i^3+\frac{m_ps^2}{EI}\right)e^{r_il}=-\frac{m_p}{EI}\left[G(s)-sf_1(l)-f_2(l)\right]-\frac{1}{s^2}\left[sf_1'''(l)+f_2'''(l)\right] \tag{89}$$

We can solve the preceding set of linear algebra equations to find c_i $(i=1,\dots,4)$ and finally deduce Y. Then by using the inverse of the Laplace transform, we get $y(x,x_0,t)$. The functional for minimizing the deformation of the beam is the same as those defined by Eqs. (67–69). The rest of procedure follows the procedure outlined earlier.

Regarding numerical simulations, for a period T of simulation, let $T=M\Delta t$. Then $t=j\Delta t$ $(j=1,\dots,M)$. For the first Δt the initial conditions are given by $f_1(x)$ and $f_2(x)$. But for the next steps $(j=2,3,\dots)$, the initial conditions are derived from the previous step; i.e., $f_1(x)$ and $f_2(x)$ are then updated in time. The same assumptions are used for Eqs. (86–89). Specifically,

$$f_1(l)|_{t=j\Delta t}=y(t,l)|_{t=(j-1)\Delta t} \tag{90}$$

$$f_2(l)|_{t=j\Delta t}=\frac{\mathrm{d}}{\mathrm{d}t}y(t,l)|_{t=(j-1)\Delta t}=\mathcal{L}^{-1}\{sY(s)_{x=l}-y(t,l)|_{t=(j-1)\Delta t}\} \tag{91}$$

$$f_1'''(l)|_{t=j\Delta}=\mathcal{L}^{-1}\{Y'''(s)\}|_{t=(j-1)\Delta t}=\mathcal{L}^{-1}\left\{\sum_{i=1}^{4}c_ie^{r_il}r_i^3+\frac{1}{s^2}\left[sf_1'''(l)|_{t=(j-1)\Delta t}+f_2'''(l)|_{t=(j-1)\Delta t}\right]\right\} \tag{92}$$

$$f_2(l)'''|_{t=j\Delta t}=\frac{\mathrm{d}}{\mathrm{d}t}y'''(t,l)|_{t=(j-1)\Delta t}=\mathcal{L}^{-1}\left\{s\sum_{i=1}^{4}c_ir_i^3e^{r_il}+\frac{1}{s}f_2'''|_{t=(j-1)\Delta t}\right\} \tag{93}$$

where $\mathcal{L}^{-1}$ denotes the inverse Laplace transform operator.

The computer algorithm is as follows:

1) Given T and l, select $\Delta t(\Delta t=T/M)$ and $\Delta x(\Delta x=l/N)$.

2) For the first Δt, find the minimized y_p by varying x_0 through its range using the finite difference gradient method of optimization.

a) Using complex linear system solving subroutine, solve for the coefficients c_i $(i=1,\ldots,r)$, and then evaluate $Y(l,x_0,s)$. Note that $c_i = c_i(s)$.

b) Using the inverse Laplace subroutine, compute $y(l,x_0,t)|_{i=1}$ for the initial conditions using the original values of $f_1(l)$ and $f_2(l)$.

3) At the kth step, using $y(l,x_0,t)|_{i=k-1}$ for $f_1(x)$ and $\partial y/\partial t$ $(l,x_0,t)|_{i=k-1}$ for $f_2(l)$, find the best $x_0(i)$ at which $y(l,x_0(i),t|_{i=k}$ is minimum. Repeat this procedure for each Δt until T is obtained.

Numerical Simulations

An illustrative example to demonstrate the theory developed is carried out for the model given by Fig. 4. The following data are considered:

$$\dot{\theta}_3 = 0; \quad \theta_1 = 0.25\pi(t+4); \quad \delta = 1.0 \text{ cm}$$

$$l_1 = l_4; \quad l_2 = l_3; \quad l = 100.0 \text{ cm}; \quad E = 2.1 \times 10^7 \text{ N/cm}^2$$

$$I = 12.2656 \text{ cm}^4; \quad m = 0.03 \text{ kg/cm}$$

$$m_b = 0.6 \text{ kg/cm}; \quad m_p = 2.0 \text{ kg}$$

The cross-sectional area of the beam is shown in Fig. 4. From the data given we can calculate the following parameters:

$$\beta = (m/EI)^{1/4}\sqrt{s} = 0.003285\sqrt{s} \tag{94}$$

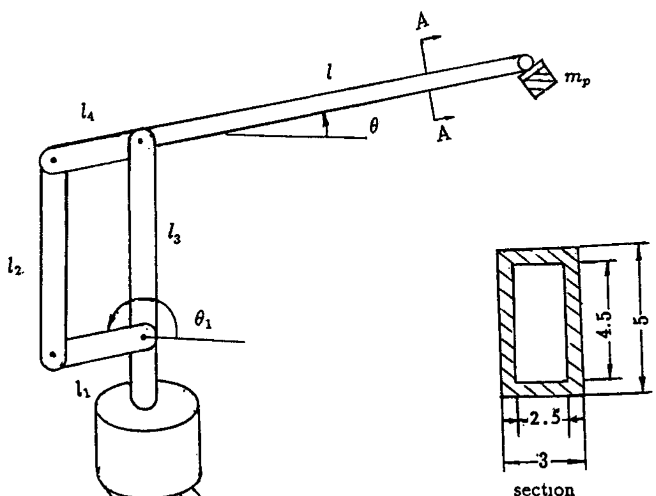

Fig. 4 Illustrative example.

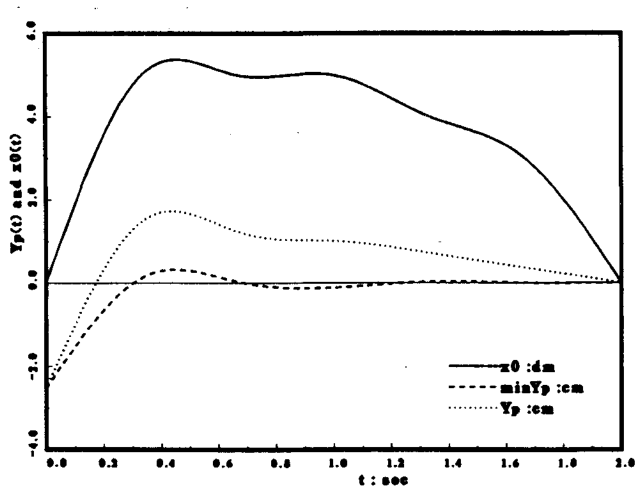

Fig. 5 Numerical simulation 1.

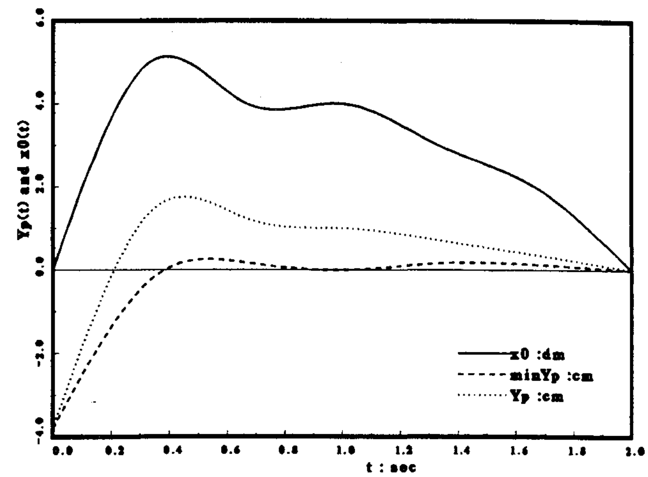

Fig. 6 Numerical simulation 2.

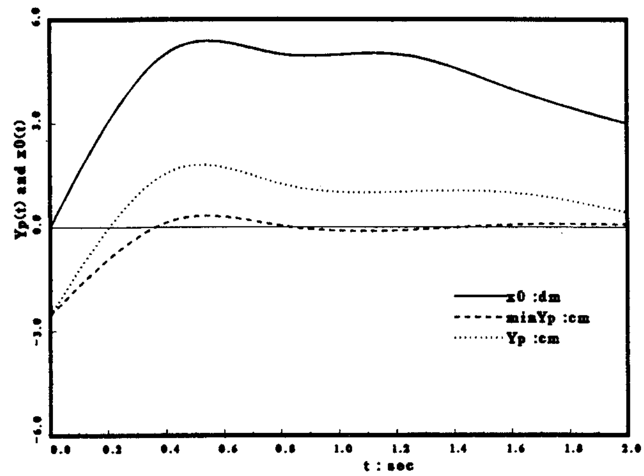

Fig. 7 Numerical simulation 3.

$$\theta = \theta_1 - \pi = 0.25\pi t \tag{95}$$

$$a_B^t = 0 \tag{96}$$

and

$$L(x,s) = mgs/(s^2 + \pi^2) \tag{97}$$

$$G(s) = gs/(s^2 + \pi^2) \tag{98}$$

The initial conditions are assumed to be

$$y(x,x_0,0) = f_1(x) = \frac{m_p g x^2}{2EI}\left(\frac{1}{3}x - l\right) \tag{99}$$

and

$$\frac{\partial}{\partial t} y(x,x_0,0) = f_2(x) = 0 \tag{100}$$

The preceding assumptions of the initial conditions state that the manipulator is in static mode $(\theta = 0)$ and the initial deformation is caused only by the load m_p. By use of the general solution we found the deformation of the beam tip to be

$$Y|_{x=l} = \sum_{i=1}^{4} c_i e^{r_i l} + \frac{1}{ms^2}\left\{L(l,s) + m\left[sf_1(l) + f_2(l)\right]\right\}$$

$$= \sum_{i=1}^{4} c_i e^{r_i l} + \frac{1}{ms^2}\left(\frac{mgs}{s^2+\pi^2} - \frac{m m_p g s x^2 l}{3EI}\right) \tag{101}$$

A numerical solution in the time domain is then sought.

Numerical results are shown in Fig. 5, where the solid curve denotes x_0 (in units of dm), the dotted curve is the deformation of the beam tip without minimization, and the dashed curve corresponds to the minimized deformation of the beam (in both cases units are in cm). The solution clearly shows how the minimization of deformation of the beam is significant when the optimization technique proposed is used. In Fig. 6 the parameters are modified where the mass is decreased; i.e., $m_p = 3.0$ kg, where the rest is kept the same. In Fig. 7, $m_p = 2.0$ kg, $\theta_1 = 0.1\pi(t+10)$. Actually, if the load m_p and the manipulator motion are kept unchanged during the cause of a particular operation, then we can optimize the mass of the mass block.

Conclusion

Two approaches for minimizing the vibration of elastic beam models with adjustable support and mass have been presented. The theory developed presents the utility of the proposed solution in the context of multibody dynamic systems and of robotic manipulators in particular. The solutions form the basis for control algorithms designed to minimize induced vibration in elastic bodies undergoing large rotations. Time-variant boundary conditions are shown to have a distinctive influence on maintaining the elastic deformation minimum.

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